Theoretically Provable Spiking Neural Networks (NeurIPS'22)

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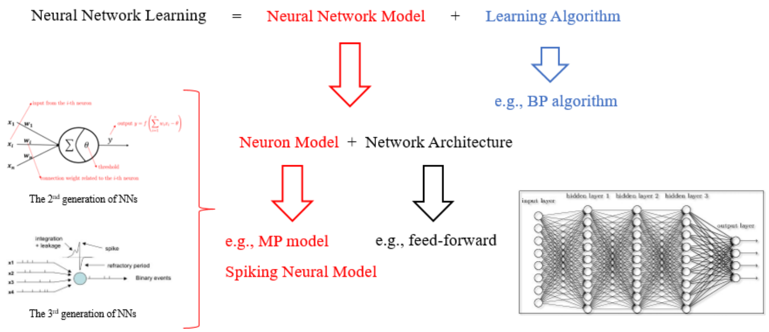
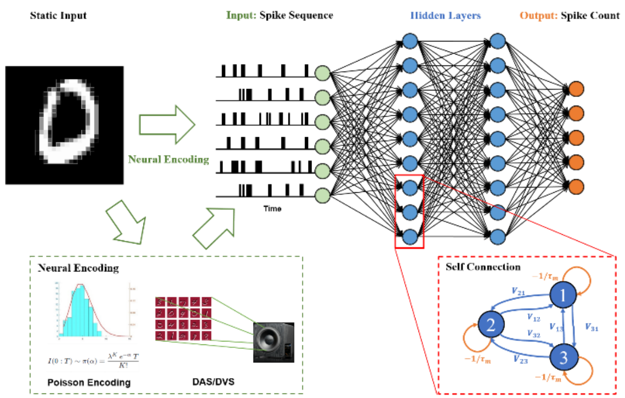
Spiking Neural Networks



SNNs take into account the time of spike firing rather than simply relying on the accumulated signal strength in conventional neural networks, and thus offering the possibility of modeling time-dependent data. The fundamental spiking neural model is usually formulated as a first-order parabolic equation with many biologically realistic (i.e., internal) hyper-parameters.

Spiking Neuron Model with Self Connections



- Formulation:

$$\frac{\partial u_k}{\partial t} = -\frac{1}{\tau_m} u_k + \sum_{j \neq k} \lambda_{kj} u_j + \frac{1}{\tau_m} \sum_{i=1}^{m} w_{ki} x_i$$

Mutual Promotion

- Reference:

Shao-Qun Zhang, Zhao-Yu Zhang, and Zhi-Hua Zhou. Bifurcation Spiking Neural Network. **Journal of Machine Learning Research (JMLR)**, 22(253):1-21. 2021.

Theoretical Investigation

- Input Poisson Sequence

$$\mathbf{I}_k(t) \sim \pi(\lambda_k)$$

- Output the Instantaneous Firing Rate (IFR)

$$f_i^{\text{ins}}(\mathbf{I}, t) = \frac{f_e(\boldsymbol{u}_i(t))}{t - t'}$$

Theorem 1 *Let $K \subset \mathbb{R}^m$ be a compact set. If the spike excitation function f_e is l-finite and $w_i \in \mathbb{R}$ where $i \in [n]$, then for all $r \in [l]$, there exists some time t such that the set of IFR functions $f(\cdot, t): K \to \mathbb{R}$ of the form $f(x,t) = \sum_{i \in [n]} w_i f_i(x,t)$ is dense in $\mathcal{C}^r(K, \mathbb{R})$.*

Universal Approximation to Discrete-time Dynamical Systems.

Theorem 2 *Given a compact set $K^m \subset \mathbb{R}^m$, a probability measure μ, and a radial function $g: K^m \to \mathbb{R}$. For some apposite spike excitation function f_e and any $\epsilon > 0$, there exists some time t such that the radial function g can be well approximated by a one-hidden-layer scSNN of $\mathcal{O}(Cm^{15/4})$ spiking neurons, that is,*

$$\| f(x,t) - g(x) \|_{L^2(\mu)} < \epsilon \,.$$

Spatial Approximation:
scSNNs asymptotically approximate a kind of radial function using a polynomial number of spiking neurons.

Theorem 3 *Let $n \geq m$. Let $x_k(t) \in \pi(\lambda_k)$ and $\mathbb{E}(x_k) = \lambda_k$ for $\lambda_k > 0$, $k \in [m]$, and $t \in [T]$. If $\mathbf{V}$ is a non-degenerate matrix, then for any $\epsilon > 0$ and matrix $\mathbf{G} \in \mathbb{R}^{m \times n}$ with $\|\mathbf{G}\|_2 < \infty$, when the time complexity satisfies*

$$T \geq \Omega\left(\frac{\sqrt{n}\, \|\mathbf{G}\|_2}{\epsilon \sqrt{\|\mathbf{V}\|_2}} \right) ,$$

there exists some one-hidden-layer scSNNs with the IFR vector $\boldsymbol{f} = (f_1, \dots, f_n)^\top$ such that

$$\| \mathbb{E}_{\boldsymbol{x}} [\mathbf{V} \boldsymbol{f}(\boldsymbol{x}, T)] - \mathbf{G}\boldsymbol{\lambda} \|_2 < \epsilon \,. \qquad (6)$$

Temporal Computation:
scSNNs converge to the linear attractors within polynomial time.